Optimal weighting estimation

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*Abstract*— After solving the homography estimation of the camera, the camera intrinsic variables and extrinsic variables can be estimated. In our case, the set of extrinsic variables are used to estimate the relative transform between tool point and camera coordinates. This means that multiple transforms are made so there must be a way to compare them. Furthermore, the relative transform between an initial and final state are known to a high precision.

# Introduction

The main idea here is to calculate weighting of different measurements based on their variances to optimize the overall weighted average variance.

# Example

In general, taking multiple measurements of a quantity and averaging the results reduces the variance of measurement. This can be shown with the simple example in Eq.1 to Eq. 4.

Suppose we let be the weighted sum of and which both have the same variance . This is basically the case when 2 measurements are taken from the same instrument in rapid succession so the system has not changed.

|  |  |
| --- | --- |
|  | (1) |
|  | (2) |
|  | (3) |

In most cases we would simply have equal weights

|  |  |
| --- | --- |
|  | (4) |

What we observe is that the variance of the weighted sum or mean average is less than that of each measurement alone. In general, if there is no covariance, there is a way to calculate the optimal weights such that the weights purely dependent on the variance of an individual measurement. The optimal weights are defined as the weights such that the overall variance reaches a minimum.

# Problem Definition

Suppose we have measurements of some quantity. We wish to calculate the weighted average so that the variance of this quantity is minimized.

Let

|  |  |
| --- | --- |
|  | (5) |
|  | (6) |
|  | (7) |

For unscaled weighted averages we have the constrain in Eq. 8.

|  |  |
| --- | --- |
|  | (8) |

The problem then become a minimization of Eq.7 subject to the constraint Eq. 8.

# Lagrange Multipliers

This problem can be easily solved using Lagrange Multipliers because the gradient of both and are extremely simple, thus creating a simple system of equations.

|  |  |
| --- | --- |
|  | (9) |
|  | (10) |

Then we have the following system of equations to solve for .

|  |  |
| --- | --- |
|  | (11) |

We observe that every weight can be written in terms of and the variance .

|  |  |
| --- | --- |
|  | (12) |

Substitution into the constrain

|  |  |
| --- | --- |
|  | (13) |

Using the properties of sums we can calculate and subsequently calculate each .

|  |  |
| --- | --- |
|  | (14) |
|  | (15) |

# Pure translation transforms

An interesting set of equations arises from calibration using pure translation and no rotation. Suppose we have 4 poses relative to the world coordinates that are all translations from one another with no rotation.

This means that the first transformation, described as a matrix is a general transformation.

|  |  |
| --- | --- |
|  | (16) |

However, the relative transform from position 1 to 2 is a translation. Between all the other positions the same is true. The relative transform is denoted by subscript .

|  |  |
| --- | --- |
|  | (17) |

Furthermore, the relative translation vector is known to good accuracy. This allows us to write the transformation as a composition of relative transforms expressed as a multiplication of transforms.

|  |  |
| --- | --- |
|  | (18) |

Note that to apply relative transforms with no rotation, the same as adding the translation vectors to .

# Weighted estimation of transofmrs

Suppose we consider the case where positions are used to calibrate the camera. We will then assume that have all be calculated via solving the homography and camera constraint system. From machine commands, we also know since the machine commanded translation is also known.

The key observation is that the rotation has not changed from to . This means there are 4 estimates of the rotation matrix which should be constant. There are also

A key point is to notice that although is augmented, after multiplication of the transform matrix, is not augmented. Another convention instead of using as the transform matrix is to expand it even further from a matrix into a matrix to keep the augmentation. This allows successive transforms to be applied very easily. An example of such a matrix is shown in Eq. 6.

|  |  |
| --- | --- |
|  | (19) |

Therefore, it follows that the augmented transformed vector can be written as the product of and .

|  |  |
| --- | --- |
|  | (20) |

It is also common for the vectors to be written as row vectors instead of column vectors. Given that we have computed already, to convert to row vectors we simply need the transpose of .

|  |  |
| --- | --- |
|  | (21) |

This allows us to easily convert conventions depending on the implementations in different libraries. Notably, Matrix3D transforms in the standard C# Media3D assembly uses the transform convention, which is why the transform must be converted if it is first obtained using the column vector convention.

# Algorithm Summary

Within the C# implementation, it is assumed that the coordinates within a checkerboard frame are known and that the corners of the checkerboard within an image can be detected with the Keyence system. That is, we assume that we have one set of points that have the locations of the corners in a coordinate system, and sets of points that correspond to the locations of the corners within each image.

The algorithm is then broken down into 4 major steps

1. Initial homography estimation
2. Solve for camera intrinsic matrix
3. Solve for a set of camera extrinsic transforms
4. Combine camera extrinsic transforms with tool-point transforms to solve tool-point to camera transform

These steps are described in detail in the following sections.

# Homography Estimation

Due to the checkerboard pattern being contained within a plane, we can define a coordinate system that makes all checkerboard corners have Z coordinate equal to zero. Originally the equation that maps 3D points onto the 2D image is in Eq. 9.

|  |  |
| --- | --- |
|  | (22) |

However, since , column of the rotation matrix can be removed for the analysis. Therefore, we can relate the vector on the left with an augmented 2D vector on the right via homography .

|  |  |
| --- | --- |
|  | (23) |
|  | (24) |

Writing in terms of its individual entries gives Eq. 12. Note that is only unique up to scale.

|  |  |
| --- | --- |
|  | (25) |

From Eq. 10 and Eq. 12, we can rewrite the equations for pixel coordinates and .

|  |  |
| --- | --- |
|  | (26) |
|  | (27) |

Let be the listed elements of in row major order.

|  |  |
| --- | --- |
|  | (28) |

Then Eq. 13 and Eq. 14 can be written as two homogeneous equations using .

Let

|  |  |
| --- | --- |
|  | (29) |
|  | (30) |
|  | (31) |
|  | (32) |

Therefore, for each pair of points and we get 2 equations. Since each point gives two equations, we can construct the matrix which concatenates all of the equations by stacking them vertically.

Matrixmust thenbe a matrix where is the number of coordinate pairs. This formulates a homogenous system of equations.

|  |  |
| --- | --- |
|  | (33) |

At this point, Singular Value Decomposition (SVD) is used to solve for the least squares solution to which is returned as the right hand singular vector associated with the smallest singular value .

# Intrinsic Estimation

The basic principle behind intrinsic estimation is that each image taken will result in 2 constraints on the intrinsic parameters of the camera. Taking a sufficient number of pictures allows the intrinsic matrix to be solved. The derivation is not shown here, but the method to construct constraint matrix to solve transformed variables , which can be used to calculate intrinsic matrix , is given.

Let be calculated as below from estimated homography .

|  |  |
| --- | --- |
|  | (34) |

Then can be formulated by stacking 2 equations for each of the images taken. Therefore, is a matrix.

|  |  |
| --- | --- |
|  | (35) |

We need at least 3 images in order to have a unique solution to up to scale.

|  |  |
| --- | --- |
|  | (36) |

The intrinsic matrix **A** is a that maps 3D points onto the 2D image pixels.

|  |  |
| --- | --- |
|  | (37) |

Each parameter in can then be calculated from. From Zhang’s paper w,e have the following formulae for the parameters which should be calculated in the same sequence.

|  |  |
| --- | --- |
|  | (38) |
|  | (39) |
|  | (40) |
|  | (41) |
|  | (42) |
|  | (43) |

At this point matrix, can be solved via substitution.

# Extrinsic Estimation

Now that is known it is a straightforward calculation to find the rotation and translation matrices defined in Eq. 9.

Let

|  |  |
| --- | --- |
|  | (44) |
|  | (45) |

Let

|  |  |
| --- | --- |
|  | (46) |
|  | (47) |
|  | (48) |
|  | (49) |

At this point, we have all the parameters to construct transformation matrix in Eq. 6.

# Combining tool point transforms

The purpose of performing this calibration method was to find the transform from the tool-point to the camera. Up to this point, the algorithm has found the rotation and translation of the camera relative to the checkerboard coordinates. In our application, the checkerboard is located in space and thus, the transform to convert a point from the checkerboard coordinates to the tool-point and vice-versa is assumed to be known.

Thus, the process is to find the aggregate transform from tool-point to checkerboard to camera times and average the transform.

Suppose we have two transforms defined by andapplied in sequence.

Then the aggregate transform has the following properties

|  |  |
| --- | --- |
|  | (50) |

Then all we must do is compute the aggregate rotation matrix and translation as in Eq. 37 to find the aggregate transform from tool-point to the camera.

# Averaging multiple transforms

The concept of averaging, in this case, means the following. Suppose is a point that we wish to transform by . Since there are estimates of we need to average them.

|  |  |
| --- | --- |
|  | (51) |

An easy method would simply be to add all the estimations of and average them.

|  |  |
| --- | --- |
|  | (52) |

This means that the averaged transformed can be thought of as adding all the transforms element by element and averaging them.

# Rotation Matrix Correction

Due to estimation error, the rotation matrix component found by averaging in Eq. 39 will not, in general, satisfy the rotation matrix. We then find the matrix that satisfies both the properties of the rotation matrix and minimizes the Frobenius norm .

|  |  |
| --- | --- |
|  | (53) |

This can be done also with the SVD. Suppose we use SVD to obtain matrices , then the expression for the best fit rotation matrix can be calculated using .

|  |  |
| --- | --- |
|  | (54) |
|  | (55) |

If the rotation matrix is improper, then use a negative left singular vector in order to computer .

# Camera calibration results

Using a phone camera, 4 images of a checkerboard were taken. This is shown in Figure 1.

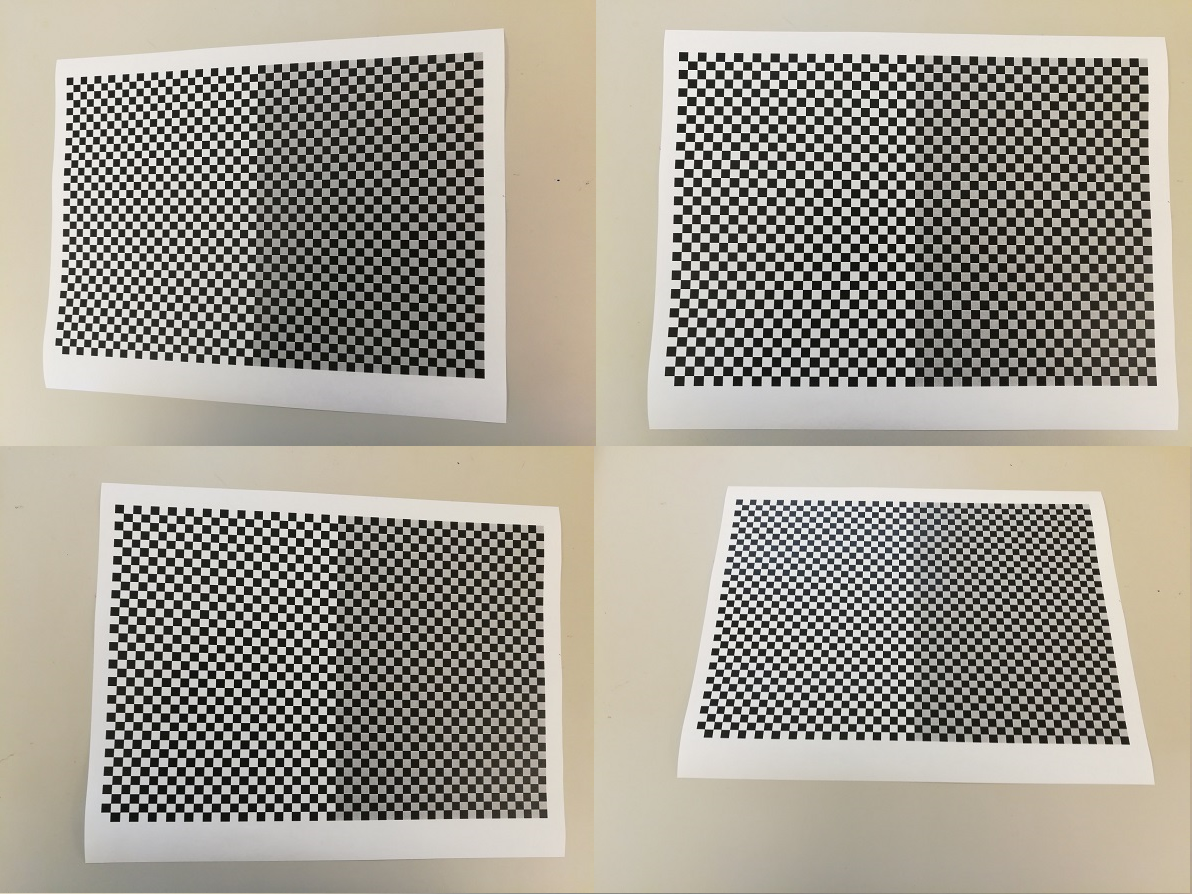


Figure 1 – All 4 images used to calibrate the camera.

Since matrices , can be calculated, we can apply the transforms on the checkerboard corners and also compare them to the image points. We simply multiply the matrices to a matrix containing the data points. This is shown in Eq, 43.

|  |  |
| --- | --- |
|  | () |

Since there is an arbitrary scale we simply divide and by the scale for each point. This gives the pixel coordinates on the image. Using this method, a reprojection was created, showing good agreement in Figure 2.

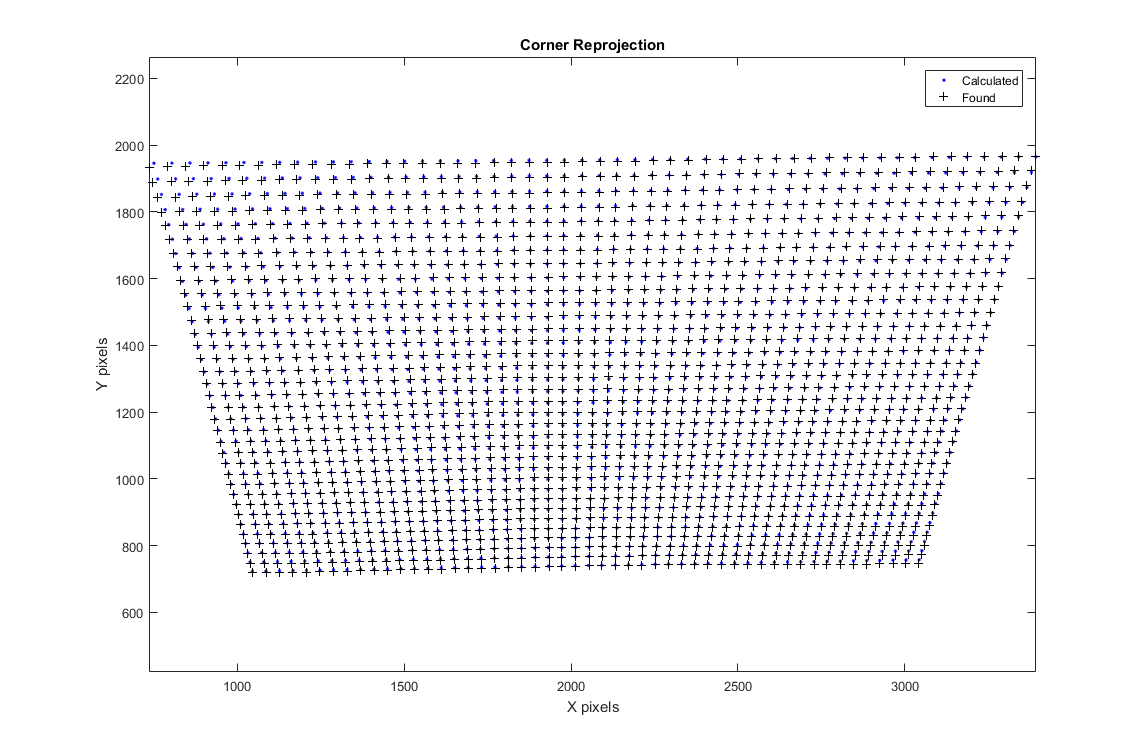


Figure 2 - Reprojection verification

# Conclusion

Although we are primarily concerned with estimating the matrix , looking at the pixel reprojection errors can give us insight on how well this model in general is performing. Thus, we considered the norm of the errors and their distribution as well as their mean. From Figure 3 we can see each pixel axis has a mean error centered around 0. In practice i,t is approximately 0.4 pixels mean error in each axis.

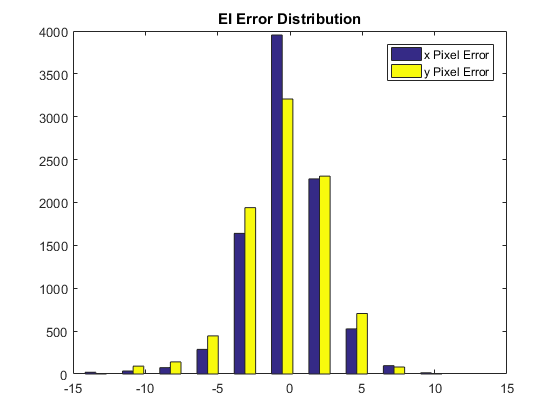


Figure 3 - Histogram of and pixel error for all 4 images

The Euclidian error is calculated shown in Eq. 46. It uses the standard deviations in each of the axes.

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|  | () |
|  | () |
|  | () |

The result is 4 pixel standard deviation error showing good agreement. This shows the validity of the calibration procedure in producing accurate results and that the transforms are valid over multiple angles.